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Considerations on the Effects Produced by Superimposed Electric and Magnetic Fields in Biological Systems and Electrolytes

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§ 1. INTRODUCTION

THE behaviour of charged particles in superimposed magnetic and electric fields or in electromagnetic fields has been the subject of numerous studies. However, practically all of this work was performed in high vacuum or in gases. Tarejew (1937) suggested, on the basis of elementary analysis, that colloidal particles undergoing electrophoresis or moving under some external force can be diverted from their path by means of a constant electric or magnetic field. Starostenko (1953) experimented with the effect of constant magnetic and electric fields in electrolytic solutions. Pressure exerted by ions on the electrolyte was measured by means of a capillary device, and it was demonstrated that there is no *Hall effect* in electrolytes. It must be pointed out that pressure measurements in a closed electrolyte cell, when current passes through the electrolytes, can be easily obscured by gas pressure generated at the electrodes. Kolin (1953) studied experimentally the movement of electrically neutral particles in constant magnetic and electric fields and the subject was subsequently analysed theoretically (Leenov and Kolin 1954). In 1943 one of us (F.H.) studied the effect of a stationary magnetic field on the movement of bacteria in a modified microelectrophoresis apparatus. It was observed that the bacteria which moved in a straight line under the influence of the electric field alone changed their direction when a magnetic field (perpendicular to the electric field) was applied, but continued to move in a straight line. Under arbitrary experimental conditions a 20° change in the direction took place. Circumstances did not permit the continuation of further experiments. The problem was considered again in 1948 and it appeared that constant electric and magnetic fields had only a limited significance since in such field conditions the primary effect of a magnetic field was a change of the direction of a moving charged object. However, it was visualized that by using synchronized alternating fields, the effect of the electric field could be eliminated (time average basis) and movement of a charged object in a viscous medium would be governed by the force component whose direction is perpendicular to the movement of object in the electric field. It was considered that perpendicular alternating fields would offer conditions where objects with positive and negative charge could be moved in the same direction (see theoretical section). Many potential uses for studies in electrolytes as well as for the effects of combined fields on biological systems could be anticipated. In order to test the validity of these considerations, an elementary theoretical analysis was carried out for sinusoidal fields and subsequently a number of experiments were carried out with alternating (60 cycle) electric and magnetic fields on the movement of ions in solutions. Recently an analysis of the subject and possible application for studies of combined field action on biological systems was reported (Heinmets 1959). In this paper we will first present some experimental data to show that extensive ionic displacements will take place in the presence of combined fields. Then a theoretical analysis, which was carried out recently, will be presented on the effect of combined fields not only on ions but also on dipoles in the solution. Dipole phenomena have not yet been studied experimentally, but on the basis of experiments with ions one can predict that measurable effects should be obtainable. Finally a discussion on the possible mechanism of the action of combined alternating fields on biological systems will be proposed.

§ 2. Experimental

2.1. *Movement of Ions in Alternating Fields*

Sixty cycle alternating magnetic and electric fields were used. Electric current for both the magnet and the electric potential to be applied at the electrodes was taken from the same source. The electrode potential passed in addition through a phase shift transformer which permitted the shifting of phase between electric and magnetic fields. A lucite trough ($1 \times 1 \times 7$ cm) served as a cell. Internal sidewalls were covered lengthwise by platinum foil (1×7 cm) and served as electrodes. The cell was placed between the magnetic poles so that electric and magnetic fields were perpendicular. Sufficient room was left between the poles so that the internal part of the cell could be observed. In order to facilitate the viewing of ionic movements coloured ions were used. The cell was filled with weak electrolyte solution (to obtain better electric field distribution) and placed between magnetic poles. The following experiments were made.

(*a*) A small grain of potassium permanganate placed at the end of the cell dissolved and formed a purple region at the bottom of the cell. When alternating electric and magnetic fields were applied separately no visible changes occurred in the cell, but when these were applied simultaneously, a visible migration of the coloured boundary could be observed lengthwise in the cell. Changing the phase of the electric potential by 180° with respect to the magnetic field caused the ions to move in the opposite direction. Similar effects were obtained when the direction of the magnetic field was reversed. The increase of either electric or magnetic field intensity produced an increase in the velocity of the boundary. A phase shift between electric and magnetic fields reduced the boundary velocity, and at 90° phase shift, the boundary became stationary. The velocity of the boundary depended on the ionic concentration at the boundary. When magnetic flux density (B) was kept constant, the following experimental velocities were observed at three arbitrary ionic concentrations, where $C_1 > C_2 > C_3$. Corresponding boundary velocities (B = 2000 gauss (peak value) ; E = 14·5 volt/cm) were 0·84, 0·61, and 0·01 cm/sec.

(*b*) In another experiment, human haemoglobin was dissolved in 0·1 M phosphate buffer at pH 9·2. A drop of haemoglobin solution was placed at one end of the cell. At E = 10 volt/cm, B = 2000 gauss, the boundary of haemoglobin moved 2·5 cm in 8 seconds (0·36 cm/sec).

Subsequent experiments revealed that salt and protein boundaries do not move visibly when movement of water is prevented. When the salt boundary is set up on a wetted filter paper or in agar-agar gel, there is no observable motion of the boundary (at the field strengths used).

The effect of the magnetic field on crystalline or fused salts could be easily observed. A concentric disc (with central hole) of solid borax was suspended in an alternating magnetic field. Platinum foil which served as electrodes was placed around the inner and outer circumference of the disc. When an alternating electric field was applied to the electrodes the salt disc started to rotate. The rotational force was of such magnitude that it could be easily measured. It depended on the magnitude of both fields, and a higher water content in the solid salt produced larger forces.

2.2. *Ionic Pump with Constant Field*

Since the water movement is associated with the movement of ions, it was decided to carry out some specific experiments. Since it was desired to carry out experiments with higher magnetic flux density, a constant field had to be used. For this particular type of experiment, where it was desired only to demonstrate a pressure head formation in an electrolyte, constant fields were adequate. Moving ions exert a viscous force (Stokes) on the water, which can be measured as a pressure head or the pressure head may be used to ' pump ' the electrolyte.

A glass trough was used as the cell (10 cm long). At the centre section, internal sidewalls were covered lengthwise with platinum electrodes (1·5 × 5 cm). The trough containing electrolyte was placed between the gap of a permanent magnet (~5000 gauss) so that the region of the electrodes was in a vertical magnetic field. When an electric potential was applied to the electrodes a difference was observed between the levels of electrolyte at each end of the trough (free of fields). When electrolyte levels were connected with a suitable hydraulic arrangement a continuous electrolyte flow could be produced through the cell. The longitudinal flow of electrolyte was easily varied by changing the current intensity across the trough. Various electrolytes were tested and they all produced a definite flow which could be measured. The most rapid flow took place when the electrolyte contained protons, but at the present it is impossible to evaluate the contributions made by proton-water molecule (H_3O^+) movement and H^+ transfer migration. Experiments suggest that there is an 'anomalous effect' by protons on water movement. Comparative flow data, as a function of current passing through the electrodes on NaCl, HCl, and NaOH are presented in Table 1. Vigorous liberation of gas took place at the electrodes but this can be avoided by application of the electric field through a suitable membrane.

Table 1. Flow of Electrolytes as a Function of Current

0·2M NaCl			0·2M HCl			0·2M NaOH		
Current mA	Fluid flow ml/min	Flow $\frac{\text{ml}}{\text{100mA-min}}$	Current mA	Fluid flow ml/min	Flow $\frac{\text{ml}}{\text{100mA-min}}$	Current mA	Cluid flow ml/min	Flow $\frac{\text{ml}}{\text{100mA-min}}$
420	18	4·3	440	32	7·3	—	—	—
620	25	4·1	760	48	6·3	600	35	5·9
770	32	4·1	—	—	—			

2.3. *General Comments*

The few preliminary experiments reported here demonstrate that in the movement of ions in combined electric and magnetic fields the forces exerted on the medium by ions are in the experimentally measurable range. It seems that by using alternating fields, the measurement of the magnetic force component may help to study many properties of liquid or solid media containing ions or charged particles. For example, in the *Hall effect* where one usually measures very low values of electric potential for good conductors, magnetic force measurements may yield more information on the system, because potential results from a *difference* of ionic mobilities while force results from the *sum* of ionic mobilities.

It must be noted that in experimenting with combined magnetic and electric fields, there is a difficulty in obtaining uniform fields. As a consequence, in the electrolytic medium, force gradients may be present and convection currents appear. In addition, when the fields are uniform but the distribution of ions is not, a similar situation may arise. As a matter of fact, it suggests that combined fields in electrolytes with suitable geometry could be used for hydrodynamic studies in creating small scale flow and convection systems.

Since the effects of electric and magnetic fields on ionized and dielectric media depend on the frequency (Schwan 1957, Brown *et al.* 1947), the experimental procedures for applying these fields to electrolytic or biological media require consideration.

§ 3. Considerations for Application of Combined Fields in Biological Studies

As was seen from previous analyses, the forces which are exerted by combined electric and magnetic fields on the ions and dipoles are of such magnitude that displacement of both species can take place, especially when water is free to move. Such a condition exists normally in extra-cellular space and, in the restricted sense, also in intra-cellular regions. What kind of metabolic and functional effects could be produced when cells are exposed temporarily or are grown in the presence of superimposed electric and magnetic fields? The problem is complex and can be answered only speculatively, but available experimental data on the movement of ions permit a rational analysis of the problem.

Elementary analysis indicates that forces present in such fields are *too small* to produce direct changes in molecules or in structural elements of the cell. We are not concerned here with the well-known heating effect of the medium, which results from oscillatory motions of the ions in an alternating electric field. How could small displacement forces (on ions and dipoles) produce any significant changes in cellular processes? Metabolic and synthetic reactions of individual units are controlled by many factors such as temperature of medium, substrate concentration at the site of reaction (at the locus of enzymes and templates), availability of nutrients, etc. In a normal environment, specific intra-cellular metabolites are distributed non-uniformly and the concentration pattern depends on the locus of enzymes and templates and many other factors. Orderly synthesis and the rate of enzyme-substrate reactions depend largely on the local substrate concentrations. At normal temperature, diffusion is the principal cause for the movement of free metabolites. It can be considered that, when in addition to thermal motion a new force is introduced into an intra-cellular region which produces displacement in metabolites, the intra-cellular concentration pattern of metabolites can be altered and completely distorted. As was seen earlier, combined electric and magnetic fields produce ion movements in solutions and it seems logical that free metabolites in ionized form will also be displaced. This displacement will depend on the strength of fields, the magnitude of ionic charge(s), the viscosity of the medium, etc. If the forces are strong, it is considered that an abnormal distribution of ionized metabolites will take place and, as a consequence, a disorganization of the normal functional pattern will follow. For example, hydrogen ions, which have an abnormal velocity, will be subjected to strong magnetic forces and may migrate to the cell boundary and so produce an abnormal pH-distribution in this region inside the cell. Since positive and negative metabolites move in the same direction, an extensive displacement of metabolites may take place. Consequently, processes such as synthesis, which require a sequential assembly of units, may be highly distorted. It appears that the effect of combined fields (at reasonable field strengths) would produce a moderate disorganization of cellular growth pattern and not necessarily a direct loss of viability. The result would be an abnormal cellular growth which in turn may lead into loss of viability.

Since experimental work on combined field action has been performed only with electrolytes, and biological effects have been postulated only on theoretical grounds (Heinmets 1959), it is gratifying to find out that recent experiments reveal distinct growth distorting effects by high frequency fields (Heller and Teixeira 1959).

§ 4. Theoretical

4.1. *The Motion of Microscopic Monopoles and Dipoles in Crossed Synchronized Electric and Magnetic Fields*

The total force experienced by a charged object moving in an electric and magnetic field is given by the well-known Lorentz expression:

$$\mathbf{F}=e(\mathscr{E}+1/c\, v\times\mathscr{B}) \qquad (1)$$

where e is the charge on the object, v its velocity, c the velocity of light and $\mathscr{E}$ and $\mathscr{B}$ the respective electric and magnetic field strengths (Gaussian units will be used throughout). The effect which we are seeking is due to the second term in (1); one might consider the process as one in which the first term causes a velocity and the second converts it into the instantaneous direction of the cross product of the fields, which, since they are synchronized, will be in constant in direction, even though the largest part of the velocity oscillates. This net motion of the particles in turn drags the fluid with it.

Besides the force given by (1), there is also the viscous drag of the medium which must be considered. This term too varies linearly with velocity (valid for relatively slow motion), and is, for an arbitrary non-uniform velocity of a spherical body:

$$\mathscr{F}=\frac{m'}{2}\frac{\mathrm{d}v}{\mathrm{d}t}+Rv+\sqrt{\left(\frac{9m'R}{4\pi}\right)}\int_{-\infty}^{t}\frac{\mathrm{d}v}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\sqrt{(t-\tau)}} \qquad (2)$$

(cf. Landau and Lifshitz 1959, p. 95). The Fourier transform of (2) is (*ibid.*):

$$\mathscr{F}=R\left[1-\frac{im'\omega}{R}+\frac{3}{2}(1-i)\sqrt{\left(\frac{m'\omega}{R}\right)}\right]\hat{\mathbf{v}} \qquad (3)$$

where we have introduced, for a sphere of radius a in a medium of viscosity η and density ρ:

$$R=6\pi a\eta;\quad m'=\frac{4\pi}{3}\rho a^3 \qquad (4)$$

as the Stokes resistance coefficient and the displaced mass respectively. The definition of the Fourier transform used is as follows:

$$f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\exp(i\omega t)\,\mathrm{d}t;\quad f(t)=\int_{-\infty}^{\infty}f(\omega)\exp(-i\omega t)\,\mathrm{d}\omega \qquad (5)$$

which in turn uses the following representation for the Dirac delta function:

$$(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\omega t)\,\mathrm{d}t. \qquad (6)$$

In terms of (1) and (2), the equation of motion for the particle of mass m in a fluid which is assumed quiescent at large distances is:

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+\mathscr{F}=\mathbf{F}. \qquad (7)$$

In the particular case of interest, the electric field is oscillatory along the x axis; the magnetic field is synchronous along the y axis (but not necessarily in phase) i.e.,

$$\begin{aligned}\mathscr{E}&=\mathscr{E}_x=\mathscr{E}_0\cos(\omega_0 t+\phi_1)\\ B&=B_y=B_0\cos(\omega_0 t+\phi_2).\end{aligned} \qquad (8)$$

It is then apparent that if the body were initially at rest, it would never achieve motion along the y direction, so that with the notation:

$$W=v_x+iV_z \qquad (9)$$

we obtain from (7) the scalar equation for the complex velocity W as:

$$\frac{1}{\gamma}\frac{\mathrm{d}W}{\mathrm{d}t}+W+\sqrt{\left(\frac{2\mu}{\pi\gamma}\right)}\int_{-\infty}^{t}\frac{\mathrm{d}W}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\sqrt{(t-\tau)}}=\alpha\cos(\omega_0 t+\phi_1)$$
$$+i\beta W\cos(\omega_0 t+\phi_2) \qquad (10)$$

where we have introduced the notation:

$$\gamma=\frac{R}{m+\frac{1}{2}m'};\quad \mu=\frac{9m'}{4(m+\frac{1}{2}m')};\quad \alpha=\frac{e\,\mathscr{E}_0}{R};\quad \beta=\frac{e\,B_0}{RC}. \qquad (11)$$

The Fourier transform of (10) is:

$$\left[1+\sqrt{\left(\frac{2\mu\omega}{i\gamma}\right)}+\frac{\omega}{i\gamma}\right]\hat{W}(\omega)=\frac{\alpha}{2}\left[(\exp(i\phi_1)\,\delta(\omega+\omega_0)+\exp(-i\phi_1)\,\delta(\omega-\omega_0)\right]$$
$$+\frac{i\beta}{2}\left[\exp(i\phi_2)\,\hat{W}(\omega+\omega_0)+\exp(-i\phi_2)\,\hat{W}(\omega-\omega_0)\right]. \qquad (12)$$

We expand the solution of (12) to first order in β since this is a small quantity, viz.

$$\beta=(8\cdot 5)10^{-8}\,B_0/a\eta$$

where a is in Ångstroms, B_0 in kilogauss and η in centipoise. The solution is then given in terms of the function:

$$\psi(\omega)=\left[1+\sqrt{\left(\frac{2\mu\omega}{i\gamma}\right)}+\frac{\omega}{i\gamma}\right]^{-1} \qquad (13)$$

in the form:

$$\hat{W}(\omega)=\frac{\alpha}{2}\left[\exp(i\phi_1)\,\delta(\omega+\omega_0)+\exp(-i\phi_1)\,\delta(\omega-\omega_0)\right]$$
$$+\frac{i\alpha\beta}{4}\left\{\exp[i(\phi_1-\phi_2)]\,\psi(\omega+\omega_0)+\exp[-i(\phi_1-\phi_2)]\,\psi(\omega)\delta(\omega)\right\}$$
$$+\frac{i\alpha\beta}{4}\left\{\exp[i(\phi_1+\phi_2)]\,\delta(\omega+2\omega_0)\psi(\omega+\omega_0)\right.$$
$$\left.+\exp[-i(\phi_2+\phi_1)]\,\delta(\omega-2\omega_0)\psi(\omega-\omega_0)\right\}\,\psi(\omega) \qquad (14)$$

which has the immediate transform:

$$W(t) = \alpha\,\mathscr{Re}\{\exp[i(\phi_1+\omega_0 t)]\ \psi(\omega_0)\}$$
$$+\frac{i\alpha\beta}{2}\mathscr{Re}\{\exp[i(\phi_2-\phi_1)]\ \psi(\omega_0)\}$$
$$+\frac{i\alpha\beta}{2}\mathscr{Re}\{\exp[i(\phi_1+\phi_2+2\omega_0 t)]\ \psi^*(\omega_0)\psi^*(2\omega_0)\} \quad . \quad . \quad (15)$$

where the symbol $\mathscr{Re}$ is used to denote the real part of a complex number. The middle term in (15) is seen to be non-oscillatory and in fact a constant, whereas the first and last terms are both oscillatory. Moreover if we average over the phase of the electric field while keeping the phase difference $\phi_0=\phi_1-\phi_2$ between the fields a constant, so as to average over the initial conditions of the particles, we see that the first and last terms vanish and we are left with a component of the velocity in the z direction only, viz.:

$$v_z = v = \tfrac{1}{2}\alpha\beta\,\mathscr{Re}[\exp\,(i\phi_0)\ \psi(\omega_0)] \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In terms of the parameters:

$$v_0=\alpha\beta=\frac{e^2\,\mathscr{E}_0\,B_0}{R^2c};\quad \tan\epsilon=\frac{\sqrt{\left(\frac{\mu\omega_0}{\gamma}\right)}+\frac{\omega_0}{\gamma}}{1+\sqrt{\left(\frac{\alpha\omega_0}{\gamma}\right)}} \quad . \quad . \quad . \quad (17)$$

this takes on the very simple form:

$$v=\tfrac{1}{2}v_0\,\frac{\sec\epsilon\ \cos\,(\phi_0+\epsilon)}{1+\sqrt{\left(\frac{\mu\omega_0}{\gamma}\right)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It is important, in order to understand the nature of this effect, to notice that the parameter ω_0/γ is extremely small for any realizable system, viz.:

$$\frac{\omega_0}{\gamma}=\frac{m\nu}{3a\eta}\Big/\left(1-\frac{2}{9}\mu\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where ν is the frequency; this is less than $10^{-10}\nu$ (ν in cycles per second) for any reasonable choice of mass and radius of the object. Thus for any system of interest, except for extremely high frequencies, we may expand (18) to give:

$$v=\tfrac{1}{2}v_0\left\{\left[1-\sqrt{\left(\frac{\mu\omega_0}{\gamma}\right)}\right]\ \cos\phi_0-\left[\sqrt{\left(\frac{\mu\omega_0}{\gamma}\right)}+(1-2\mu)\frac{\omega_0}{\gamma}\right]\ \sin\phi_0\right\}. \quad (20)$$

In the case of ions one may neglect μ, but in that case the long range Coulomb forces between ions creates a larger ionic atmosphere characterized by the Debye–Hüeckel radius, d,

$$\frac{1}{\mathrm{d}^2}=\frac{4\pi\mathrm{e}^2}{kT}\sum n_i Z_i^{\,2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

(Landau and Lifshitz 1958, p. 231), which becomes small for large concentrations (n is the number of ions per unit volume and Z is the ionic valence). A 0·275 molar concentration gives $d = 1$ Å at room temperature. The effect of the ionic atmosphere introduces other moments into the force equation, (1) and changes the effective viscosity (2). For the case of alternating electric fields only, the problem has been solved by Falkenhagen 1934. As far as we know; this has not been done anywhere for the case of the combined fields. However, it will not be done here because of lack of space and also, because no such effect (ionic atmosphere) exists for forces which fall off faster than the square of the distance, e.g. in the case of dipoles, which is our main concern.

We had assumed earlier that the fluid is quiescent at large distances from the charged body; obviously this cannot remain the case because of the reaction to the viscous drag (2). If we sum the force over all the ions in a unit volume, this is equivalent to averaging over initial conditions (ϕ_1) and multiplying by n the number of ions per unit volume, we obtain the force per unit volume in the fluid, which is the same as a pressure gradient, i.e.

$$6\pi a\eta n(v-u) = \nabla p \qquad \text{(21a)}$$

where u is the mean fluid velocity at a point in the volume. Because of (18) this is seen to be independent of the time and along the z direction. The solution of the relevant Navier-Stokes equation for the fluid, for uniform flow, is straightforward. One should note, at this time, that to lowest order in ω_0/γ it would be the difference $(v-u)$ which would be on the left side of (20a)), being identically the same as that for Poiseuille flow, in lowers order in ω_0/γ. For a pressure gradient of:

$$\nabla p = 3\pi a\eta n v_0 \cos\phi_0 = \frac{ne^2}{12\pi a\eta c}\, \mathscr{E}_0\, B_0 \cos\phi_0 \qquad \text{(22)}$$

one obtains a mean velocity across a pipe of radius r of:

$$\bar{u} = \frac{ne^2}{96\pi a\eta^2 c}(\mathscr{E}_0\, B_0 \cos\phi_0)r^2 \qquad \text{(23)}$$

(Landau and Lifshitz 1959, p. 57). If one puts a baffle in the pipe, or uses the lower part of a manometer for the pipe, one can measure the pressure gradient (22) directly. In any case, for not too high frequencies, this is a convenient measurement of particle sizes, and becomes particularly important in the case of dipoles where the ordinary estimate by means of electrical mobilities becomes meaningless.

To make the preceding numerical results more tractable, we shall introduce the following notation:

$\mathscr{E}$ will be measured in e.s.u. (300 volts cm)
B will be measured in kilogauss (10^3 gauss)
a will be measured in Ångstrom units (10^{-8} cm)
n will be measured in moles per litre (molar concentration) . (24)
m will be measured in molecular weights
η will be measured in centipoise (10^{-2} poise).

With this particular choice, the relevant equations are:

$$\gamma = \frac{6\pi a\eta}{m+\frac{1}{2}m'} = (1\cdot14)10^{15}\frac{a\eta}{m}\left(1-\frac{2}{9}\mu\right)\Big/\text{sec};\quad \mu = \frac{4\cdot5}{1+0\cdot79\ m/\rho a^3} \quad . \quad (25)$$

$$v_0 = \frac{e^2\,\mathscr{E}_0\,B_0}{R^2C} = (2\cdot16)10^{-9}\,\frac{\mathscr{E}_0\,B_0}{a^2\eta^2}\ \text{cm/sec} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$h = \frac{l\nabla p}{\rho g} = (1\cdot25)\left(\frac{nl}{a\rho\eta}\,\mathscr{E}_0\,B_0\,\cos\phi_0\right)\text{cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The method of the manometer tube, in which the net force on the fluid is zero so that the fluid velocity is zero (on the average) has advantages in that the problem of ions of opposite charge diffusing through each other's atmospheres does not occur. Ions of both signs move in the same direction (that of the cross product of the fields). The effective ionic radius can be found from (27) by measuring the difference in heights between the two manometer arms and there is no reason why large concentrations cannot be used. One caution must be given, that an appreciable concentration gradient should be built up in time due to the steady drift of ions through the fluid as indicated by (26)—for a long enough run. This will ultimately come into equilibrium with the thermal diffusion, making a uniform concentration gradient.

A dipole, taken as a whole, is electrically neutral; however, an electric field will produce a torque on it and tend to rotate it so that its dipole moment ($\boldsymbol{p}$) is parallel to the field, the magnitude of the torque is:

$$\boldsymbol{L} = \boldsymbol{p}\times\mathscr{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Once the dipole begins to rotate (with an angular velocity Ω), its (vector) dipole moment changes as follows:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \equiv \dot{\boldsymbol{p}} = \Omega\times\boldsymbol{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and a dipole with a changing dipole moment experiences a force in a magnetic field which is:

$$\boldsymbol{F} = \frac{1}{c}\dot{\boldsymbol{p}}\times\boldsymbol{B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If we now agree to limit ourselves to frequencies very much less than γ (25), all inertial effects can be neglected as well as those due to the fluctuation of the boundary layer so that the viscosity terms for rotation as well as translation are of the ordinary Stokes type, i.e.

$$\mathscr{L} = R'\Omega = 8\pi a^3\eta\Omega;\quad \mathscr{F} = Rv. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Combining (28), (30) and (31), we obtain the equations of motion:

$$R'\Omega = \boldsymbol{p}\times\mathscr{E};\quad Rv = \frac{1}{c}\dot{\boldsymbol{p}}\times\boldsymbol{B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

One readily finds by choosing our directions as in (8) that if we choose θ as the angle between the x axis and the instantaneous direction of

$\boldsymbol{p}$ and ϕ as the angle which the projection of $\boldsymbol{p}$ on the y–z plane makes with the y axis, that ϕ is a constant and θ satisfies the following equation:

$$\dot{\theta} \operatorname{cosec} \theta = -\frac{p\mathscr{E}}{R'} = -\frac{p\mathscr{E}_0}{R'} \cos(\omega t + \phi_1). \quad . \quad . \quad . \quad . \quad (33)$$

Also the components of Ω are:

$$\Omega_x = 0;\ \Omega_y + i\Omega_z = i\dot{\theta} \exp(i\phi). \quad . \quad . \quad . \quad . \quad (34)$$

The integral of (33) is most easily expressed in terms of the parameter:

$$\chi = \int_0^t \frac{p\mathscr{E}}{R'} \mathrm{d}t = \frac{p\mathscr{E}_0}{R'\omega} [\sin(\omega t + \phi_1) - \sin \phi_1] \quad . \quad . \quad . \quad (35)$$

so that one obtains:

$$\cos \theta = \frac{1 + \cos \theta_0 \coth x}{\coth x + \cos \theta_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Introducing the notation of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as unit vectors in the x, y, z directions, we have put:

$$\boldsymbol{p} = p\{\boldsymbol{i} \cos \theta + \boldsymbol{j} \sin \theta \cos \phi + \boldsymbol{k} \sin \theta \sin \phi\} \quad . \quad . \quad (37)$$

so that from (33) we obtain:

$$\boldsymbol{v} = \frac{pB}{RC}\left\{\boldsymbol{k} \frac{\mathrm{d}}{\mathrm{d}t} \cos \theta - \sin \phi\, \boldsymbol{i} \frac{\mathrm{d}}{\mathrm{d}t} \sin \theta\right\}. \quad . \quad . \quad . \quad (38)$$

Now, as in the treatment of monopoles, we are interested in the motion of a large number of particles and hence, in the motion as averaged over initial conditions. In this case this consists of two parts, one is the average over initial directions and the second the average over the initial phase of the field. The average of (38) over initial directions of $\boldsymbol{p}$ gives:

$$\langle \boldsymbol{v} \rangle_{\measuredangle} = \boldsymbol{k} \frac{pB}{RC} \frac{d}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}x} (x \coth x) = \boldsymbol{k} \frac{p^2 \mathscr{E} B}{RR'C} \frac{\mathrm{d}^2}{\mathrm{d}x^2} (x \coth x) \quad . \quad . \quad (39)$$

since the average of the $\sin \phi$ gives zero and the average of the $\cos \theta$ gives:

$$\langle \cos \theta \rangle = \coth x - x \operatorname{cosech}^2 x = \frac{\mathrm{d}}{\mathrm{d}x} (x \coth x) \quad . \quad . \quad (40)$$

and is an operation which commutes with the time derivative.

Introducing the notation:

$$\lambda = \frac{2p\mathscr{E}_0}{R'\omega};\quad \epsilon = \phi_1 + \frac{\omega t}{2};\quad y = \sin \frac{\omega t}{2};\quad v_0' = \frac{p^2\, \mathscr{E}_0\, B_0}{RR'C} \quad . \quad . \quad (41)$$

and also:

$$x = \lambda y \cos\left(\frac{\omega t}{2} + \phi_1\right) = \lambda y \cos \epsilon$$

$$f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x \coth x) = 2(\coth^2 x - 1)(x \coth x - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

we can put:

$$\langle v \rangle_{\measuredangle} = \langle v_\zeta \rangle_{\measuredangle} = \tfrac{1}{2} v_0' [\cos \phi_0 + \cos(\omega t + \phi_0 + 2\epsilon)] f(x) \quad . \quad . \quad . \quad (43)$$

The complete average over initial conditions of (38) consists of the average of (43) over the initial phases of ϕ_1 keeping ϕ_0 constant. This is obviously:

$$\langle v\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle v\rangle_{\measuredangle}\mathrm{d}\phi_1 = \frac{1}{2\pi}\int_0^{\pi}[\langle v(\epsilon)\rangle_{\measuredangle} + \langle v(-\epsilon)\rangle_{\measuredangle}]\mathrm{d}\epsilon. \quad . \quad (44)$$

To perform the operation in (44), we need the following representation for the coth x and consequently for $f(x)$,

$$x \coth x = 1 + 2\sum_{n=1}^{\infty}\frac{x^2}{x^2+n^2\pi^2}$$

$$f(x) = 2\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\sum\frac{\lambda^2}{n^2\pi^2+\lambda^2y^2\cos^2\epsilon} \quad . \quad . \quad . \quad . \quad (45)$$

(Wilson 1912, p.454). Again since the operations are linear and independent, they commute, so that we may write:

$$\langle v\rangle = \frac{iv_0'}{\pi}\sum_1^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\lambda^2\int_0^{\pi}\frac{\cos\phi_0+\cos(\omega t+\phi_0)\cos\epsilon}{(2n^2\pi^2+\lambda^2y^2)+\lambda^2y^2\cos\epsilon}\mathrm{d}\epsilon. \quad . \quad . \quad (46)$$

The integrals are straightforward and after some reduction, we obtain:

$$\langle v\rangle = iv_0'\sum_1^{\infty}\frac{n\pi}{(n^2\pi^2+\lambda^2y^2)^{5/2}}\left[(n^2\pi^2+\lambda^2y^2)y\sin\left(\frac{wt}{2}+\phi_0\right)\right.$$
$$\left.-(2\lambda^2y^2-n^2\pi^2)\cos\frac{\omega t}{2}\cos\left(\frac{\omega t}{2}+\phi_0\right)\right]. \quad . \quad . \quad (47)$$

Using the following integral of the Bessel function (Magnus and Oberhettinger 1954, p. 33)

$$\int_0^{\infty}\mathrm{e}^{-au}uJ_0(bu)\mathrm{d}u = \frac{a}{(a^2+b^2)^{3/2}} \quad . \quad . \quad . \quad (48)$$

the summation and integration order may be reversed, so that one may finally obtain:

$$\langle v\rangle = 2v_0'\int_0^{\infty}\frac{u\,\mathrm{d}u}{\mathrm{e}^{\pi u}-1}\left[J_0(u\lambda y)\cos\phi_0-(u\lambda y)J_1(u\lambda y)\right.$$
$$\left.\cos\frac{\omega t}{2}\cos\left(\frac{\omega t}{2}+\phi_0\right)\right] \quad . \quad . \quad . \quad . \quad (49)$$

In performing the angular average to obtain (39), we deliberately ignored the Boltzmann weighting factor which involves the energy of the dipole in the electric field ($\mathbf{p}\,.\,\mathscr{E}$) on the ground that we shall limit ourselves to fields much less than kT/p. Since kT at room temperature (300°) is $(4{\cdot}14)10^{-14}$ erg, the following table of dipole moments of common substances (Smithsonian Physical Tables #456) indicates that this is not a stringent requirement (in Debye units$=10^{-18}$ e.s.u.):

Table 2

NH_3	1·46	KCl	6·3	CHN	2·94
DCl	1·089	NaI	4·9	CH_2O	2·27
HF	1·91	SO_2	1·7	CH_3Cl	1·86
HI	0·38	H_2O	1·84	CH_4O	1·69

and we will agree to use relatively weak (less than 10^2 e.s.u.) fields. If we also do not use extremely low frequencies, the parameter λ (41) will be relatively small; e.g. a reasonable experiment would be to dissolve an extremely polar compound in a non-polar liquid in which it will not ionize and perform the manometer experiment with this solution. Several such liquids with their viscosities (room temperature) in centipoise are given in table 3 (Smithsonian Tables #334):

Table 3

CCl_4	0·845	C_6H_6	0·566	C_6H_{14}	0·296
CS_2	0·352	C_5H_{12}	0·220	H_2O	0·855

Thus, using the convention in (24), we may estimate λ as: $(1{\cdot}27)10^4$ $p\mathscr{E}_0/a^3\eta\nu$ (with p in Debye units), so that if we use fields of a few volts per cm and close to a megacycle frequency we will have small λ. One caution must be exercized, however, that with megacycle frequencies, the inertial effects are still negligible but the fluctuation in the boundary layer due to the oscillations are not, and the simple method which has been introduced so far is not adequate. The other extreme of low frequencies and high fields means very large λ.

For small values of λ, we may expand the Bessel functions in (49) and integrate term by term, using the representation for the Bernoulli numbers (Whittaker and Watson 1952, p. 126):

$$B_n = 4n\int_0^\infty \frac{u^{2n-1}\,du}{e^{2\pi u}-1} = \frac{1}{6}, \frac{1}{30}, \frac{1}{42} \cdots \qquad (50)$$

to obtain the following series in λ for the velocity:

$$\langle v\rangle = \frac{1}{3}v_0'\left\{\cos\phi_0 - \frac{1}{5}\lambda^2y^2\left[\frac{1}{2}\cos\phi_0 + \cos\frac{\omega t}{2}\cos\left(\frac{\omega t}{2}+\phi_0\right)\right]\right.$$
$$\left. + \frac{1}{21}\lambda^4y^4\left[\frac{1}{4}\cos\phi_0 + \cos\frac{\omega t}{2}\cos\left(\frac{\omega t}{2}+\phi_0\right)\right]\cdots\right\} \qquad (51)$$

It is seen that for very small λ, one has a uniform velocity with a superimposed small oscillation.

What is of more interest than the average dipole velocity as a function of the time is the average of this over an entire period. Using the following theorem (Magnus and Oberhettinger 1954, p. 28):

$$\frac{2}{\pi}\int_0^{\pi/2} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\,d\theta = J_\mu(z)\,J_\nu(z) \qquad (52)$$

one readily obtains:

$$v = 8v_0'\cos\phi_0\int_0^\infty \frac{J_0^2(\lambda u)-J_1^2(\lambda u)}{e^{2\pi u}-1}\,u\,du \qquad (53)$$

which gives, for small λ:

$$v = \frac{1}{3}v_0'\cos\phi_0\left(1-\frac{3}{40}\lambda^2+\frac{5}{672}\lambda^4+\ldots\right). \qquad (54)$$

One sees that for small λ (see fig. 2) the relevant quantity for comparison between the dipole and monopole cases is v_0' to v_0, i.e. the ratio:

$$\frac{v_0'/3}{v_0/2}=\frac{p^2}{2a^2e^2}=(2{\cdot}18)10^{-2}\ p^2/a^2 \qquad (55)$$

where the numerical result involves the convention (24). The results are seen to be comparable. The pressure 'head' analogous to (27) for the dipole case is readily found to be:

$$h=\frac{6\pi a\eta\bar{v}nl}{\rho g}=(2{\cdot}71)10^{-2}\left(\frac{nlp^2}{a^3\rho\eta}\mathscr{E}_0\ B_0\ \cos\phi_0\right)\text{ cm.} \qquad (56)$$

Although we have used, with impunity, the Stokes resistance parameter (31) valid only for rotating spheres, we may take the dependence on a in (56) as a definition of the effective radius of the dipole, i.e. as defining the high frequency combined electric and magnetic mobility of the dipole. This becomes more meaningful for larger objects than molecules, when the expansion also becomes better because λ is smaller.

The case for large λ is most tractable in the form of the series representation in (46); one may either start with (46) directly or revert to it by means of (52) through expansion of the denominator in (53), in any case obtaining, after appropriate manipulation:

$$v=\frac{8v_0'\ \cos\phi_0}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi/2}\frac{n\pi\ \cos^2\theta\ d\theta}{(n^2\pi^2+\lambda^2\ \cos^2\theta)^{3/2}}. \qquad (57)$$

The integrals in (57) are elliptic and one may evaluate them explicitly (notation for elliptic integrals is that of Jahnke and Emde 1945, p. 73) and write:

$$\bar{v}=\frac{8}{\pi}v_0'\ \cos\phi_0\sum_1\frac{n\pi}{(n^2\pi^2+\lambda^2)^{3/2}}D\left[\frac{\lambda}{\sqrt{(n^2\pi^2+\lambda^2)}}\right]. \qquad (58)$$

However, this result is no more transparent than (57) directly. In either case, one might expand for small λ and tediously rearrange terms and obtain the series (54), but for large λ it is more convenient to convert the sum over n directly into an integral by the Euler–Maclaurin sum rule (Whittaker and Watson 1952, p. 127):

$$\sum_{n=1}^{\infty}f(n)=\int_1^{\infty}f(x)\mathrm{d}x+\tfrac{1}{2}[f(1)+f(\infty)]+\sum_{m=1}^{\infty}(-1)^m\frac{Bm}{(2m)!}[f(1)^{(2m-1)}-f(\infty)^{(2m-1)}]. \qquad (59)$$

This gives us the following asymptotic series for the mean velocity:

$$\bar{v}=8\frac{v_0'\ \cos\phi_0}{\pi^2}\left\{\frac{1}{\sqrt{(\pi^2+\lambda^2)}}B\left[\frac{\lambda}{\sqrt{(\pi^2+\lambda^2)}}\right]+\frac{5\pi^2}{12(\pi^2+\lambda^2)^{3/2}}D\left[\frac{\lambda}{\sqrt{(\pi^2+\lambda^2)}}\right]+\dots\right\} \qquad (60)$$

which for λ much larger than π becomes (Jahnke & Emde 1945) :

$$\bar{v}=\frac{8v_0'\ \cos\phi_0}{\pi^2\lambda}\left\{1+\frac{\pi^2}{12\lambda^2}\left(109\frac{\lambda}{4\pi}-2\right)+\dots\right\}. \qquad (61)$$

The leading term is seen to be independent of $\mathscr{E}$ so that for very large λ (see fig. 2) the velocity depends only on the phasing between the fields and the magnetic field. If we multiply (61) by the period, $2\pi/\omega$, we find the total displacement in a cycle due to the leading term to be (in the usual convention):

$$z=\frac{8p\ B_0 \cos \phi_0}{\pi RC}=(4{\cdot}5)10^{-16}\frac{pB_0}{a\eta}\text{ cm.} \quad . \quad . \quad . \quad . \quad (62)$$

This differs from the total displacement for a dipole in d.c. fields by the factor $8 \cos \phi_0/\pi$, and in any case is seen to be quite negligible.

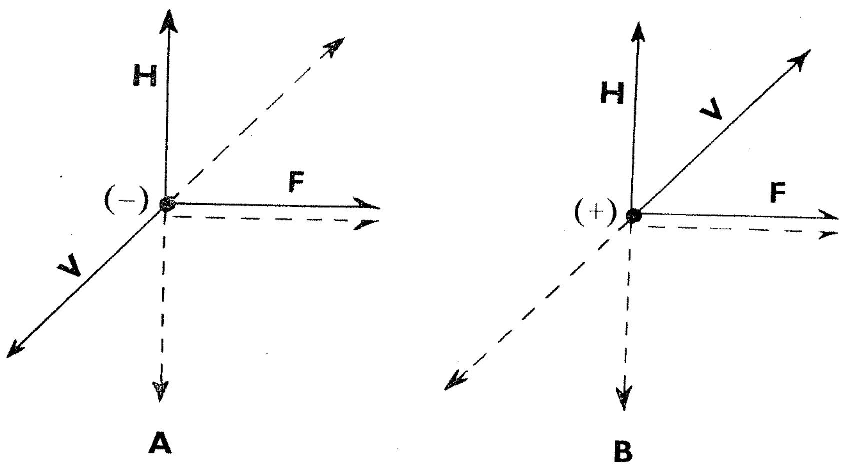

Fig. 1. Positive and negative ions, which move perpendicular to the magnetic field, are subject to the magnetic force which is perpendicular to the direction of the magnetic field and the direction of movement. Under the influence of the electric field, where positive (B) and negative (A) ions move in the opposite directions, the force exerted by the magnetic field is at the same direction for positive and negative ions. When the directions of ionic movements and magnetic fields are reversed simultaneously, the direction of magnetic force is unaltered.

H—direction of magnetic field.
V—direction of ionic movement.
F—direction of magnetic force exerted on the moving ion.

In order to obtain a measurable pressure head and thereby find the effective mobility of the dipoles under the effect of combined synchronous fields, one must balance out the two effects of maximizing h and yet keep λ small, since $\bar{v}$ is obviously a decreasing function of λ. One parameter which enters into h but not into λ can be increased with impunity, namely the concentration. Since the dipole-dipole interaction falls off more rapidly than the square of the distance, no atmospheric problems can arise until the inter-particle spacing becomes comparable with their sizes, i.e. until the solution becomes disproportionately solute.

Further analysis is needed on other shapes besides spheres and on the effect of the moving boundary layer due to the oscillations on the viscosity before one may apply these results to arbitrarily high frequencies and larger objects.

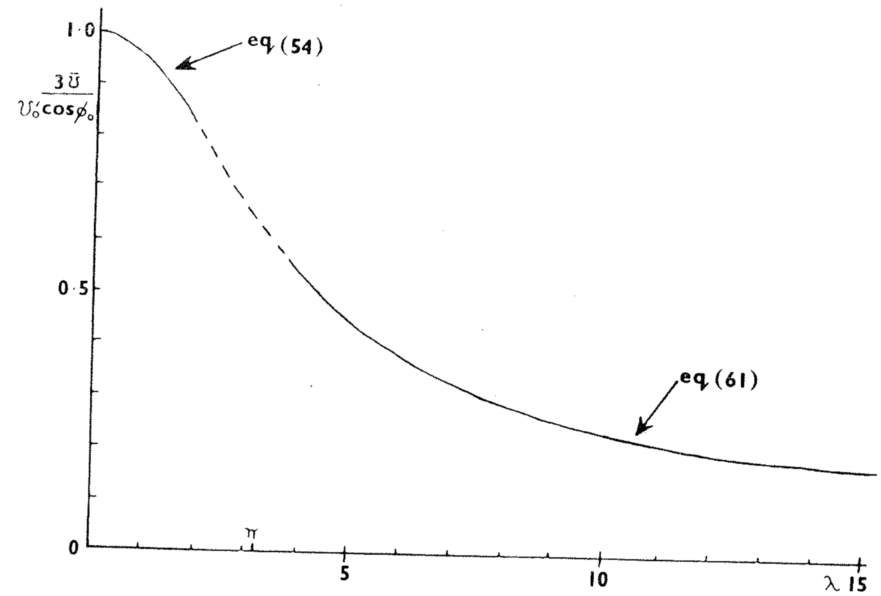

Fig. 2. Velocity of dipole as a function of the parameter λ.

Another effect which might be briefly considered here is the motion of induced dipoles under the effect of these fields. We here set α equal to the polarizability of the object so that:

$$\boldsymbol{p} = \alpha\mathscr{E} = \alpha\mathscr{E}_0 \cos(\omega t + \phi_1). \qquad (63)$$

Then from (32) we obtain the velocity as:

$$\boldsymbol{v} = \frac{1}{RC}\dot{\boldsymbol{p}} \times \boldsymbol{B} = -\frac{\alpha\omega}{RC}(\mathscr{E} \times \boldsymbol{B}_0) \sin(\omega t + \phi_1) \cos(\omega t + \phi_2) \qquad (64)$$

and the average over the initial phase is:

$$\bar{v} = \bar{v}_z = \frac{\alpha\omega\, \mathscr{E}_0\, B_0}{2RC} \sin\phi_0 \qquad (65)$$

which is seen to be independent of the time and maximal for fields which are completely out of phase. It is also much smaller than the low frequency monopole velocity, having the ratio (we use the convention in (24) plus the fact that α is comparable with the particle volume and hence is measured in Å^3) of (65) to (20) as:

$$\frac{\alpha\omega R}{e^2} \tan\phi_0 \approx (0{\cdot}51)10^{-14}\,(\alpha\nu a\eta). \qquad (66)$$

The more general case of a tensor polarizability involves a more complete analysis of the effect of shape—since this is the usual cause of departure of α from a scalar. This problem is of considerable interest, particularly in that it more directly relates to such larger objects as micro-organisms and macro-molecules under the effects of these fields and is being looked into.

§ 5. General Comments

It should be emphasized, that if biological media or electrolytes are subjected to the combined field action, the method of application is highly significant. Theoretical analysis shows that the strongest interaction forces are obtained when independent but synchronized magnetic and electric fields in the same phase and in proper geometric relation are applied. On the other hand it should be noted, that by application of a high frequency electric field alone, there is present also a magnetic induction field, but this is essentially out of phase (depending essentially on energy losses in the system) and the force component is small. An essentially similar case appears when a high frequency magnetic field is applied. Sometimes the application of a single high frequency field may be under some experimental conditions preferable, even when the force component is small. It should be noted that strong electric fields cause a large energy loss in the system and consequently a large temperature rise will take place. This can be partially reduced by application of high frequency fields in short impulses.

It is further noted that electric and magnetic fields of a travelling wave are in phase and perpendicular to each other, which are the conditions where the strongest interaction force on an ionized particle in a liquid medium is expected. However, use of such fields is markedly limited due to the dielectric heating of the medium or object under consideration.

Summary

The total force experienced by a charged object moving in an electric and magnetic field is given by the well-known Lorentz expression:

$$\mathbf{F}=e\,\mathbf{E}+\frac{1}{c}\boldsymbol{v}.\times\mathbf{B})$$

where e is the charge on the object, $\boldsymbol{v}$ its velocity, c the velocity of light and $\mathbf{E}$ and $\mathbf{B}$ the respective electric and magnetic field strengths. The effects analysed are due to the second term in equation in case where fields are crossed and synchronized. The motion of microscopic monopoles and dipoles in a liquid medium are determined by the theoretical calculations. Elementary experimental data are presented to demonstrate ionic movement and associated water movement in alternating crossed fields.

A speculative analysis of the effect of crossed fields on cellular function and growth is presented.

Résumé

La force totale éprouvée par un objet chargé qui se déplace dans un champ électrique et magnétique est donnée par l'expression de Lorentz bien connue

$$\mathbf{F} = e(\mathbf{E} + \frac{1}{c}\boldsymbol{v} \times \mathbf{B})$$

où e est la charge de l'objet, $\boldsymbol{v}$—sa vélocité, c—la vélocité de la lumière, et $\mathbf{E}$ et $\mathbf{B}$ sont respectivement les intensités du champ électrique et magnétique. Les effets analysés sont causés par le deuxième terme de l'équation pour le cas des champs croisés et synchronisés. Le mouvement des monopoles et dipoles microscopiques dans un milieu liquide est determiné par des calculations théoriques. On présente des résultats expérimentaux élémentaires afin de démontrer le mouvement d'ions et le mouvement d'eau simultané dans les champs croisés alternants.

On présente une analyse spéculative relative à l'effet des champs croisés sur la fonction cellulaire et la croissance.

Zusammenfassung

Die von einem im elektrischen und magnetischen Feld sich bewegenden geladenen Körper empfundene Kraft wird durch den bekannten Lorentz'schen Ausdruck

$$\mathbf{F} = e(\mathbf{E} + \frac{1}{c}\boldsymbol{v} \times \mathbf{B})$$

gegeben, wo e die Ladung auf dem Körper bedeutet, $\boldsymbol{v}$—seine Geschwindigkeit, c—die Lichtgeschwindigkeit, und $\mathbf{E}$ und $\mathbf{B}$—die elektrische und magnetische Feldstärke. Die analysierten Effekte sind dem zweiten Glied der Gleichung zuzuschreiben, falls die Felder gekreuzt und synchronisiert sind. Die Bewegung der mikroskopischen Monopole und Dipole in flüssigem Medium wird mittels theoretischer Berechnungen bestimmt. Es werden elementare experimentelle Ergebnisse dargestellt, damit die Ionenbewegung und die zugehörige Bewegung des Wassers in alternierenden gekreuzten Feldern demonstriert werden kann.

Es wird eine spekulative Analyse des Effektes der gekreuzten Felder auf Zellfunktion und Wachstum beschrieben.

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